Example 8-4

The rotor shaft of a helicopter drives the rotor blades that provide the lifting force to support the helicopter in the air (Fig. 8-24a). As a consequence, the shaft is subjected to a combination of torsion and axial loading (Fig. 8-24b).

For a 50-mm diameter shaft transmitting a torque $T = 2,4\, kN.m$ and a tensile force $P = 125kN$, determine the maximum shear stress in the shaft.

![Fig. 8-24 Example 8-4. Rotor shaft of a helicopter (combined torsion and axial force).](image)

**Solution**

The stress in the rotor shaft are produced by the combined action of the axial force $P$ and the torque $T$ (Fig. 8-24b). Therefore, the stress at any point on the surface of the shaft consist of a tensile stress $\sigma_0$ and shear stresses $\tau_0$ as shown on the stress element of Fig. 8-24c. Note that the $y$ axis is parallel to the longitudinal axis of the shaft.

The tensile stress $\sigma_0$ equals the axial force divided by the cross-sectional area:

$$\sigma_0 = \frac{P}{A} = \frac{4P}{\pi d^2} = \frac{4(125\, kN)}{\pi (50\, mm)^2} = 63.66\, MPa$$

The shear stress $\tau_0$ is obtained from the torsion formula (Eq. 3-11 of Section 3.3):

$$\tau_0 = \frac{Tr}{Ip} = \frac{16T}{\pi d^3} = \frac{16(2.4\, kN.m)}{\pi (50\, mm)^3} = 97.78\, MPa$$

The stress $\sigma_0$ and $\tau_0$ act directly on cross sections of the shaft.
Knowing the stress $\sigma_0$ and $\tau_0$, we can now obtain the principal stresses and maximum shear stresses by the methods described in section 7.3. The principal stresses are obtained from Eq. (7-17):

$$
\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{d}
$$

Substituting $\sigma_x = 0$, $\sigma_y = 63.66$ MPa, and $\tau_{xy} = -\tau_0 = -97.78$ MPa, we get

$$
\sigma_{1,2} = 32 \text{ MPa} \pm 103 \text{ MPa}
$$

or

$$\sigma_1 = 135 \text{ MPa} \quad \sigma_2 = -71 \text{ MPa}
$$

These are the maximum tensile and compressive stress in the rotor shaft.

The maximum in-plane shear stresses (Eq. 7-5) are

$$
\tau_{\text{max}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{e}
$$

This term was evaluated above, so we see immediately that

$$\tau_{\text{max}} = 103 \text{ MPa}
$$

Because the principal stresses $\sigma_1$ and $\sigma_2$ have opposite signs, the maximum in-plane shear stresses are larger than the maximum out-of-plane shear stresses (see Eqs. 7-28a, b, and c and the accompanying discussion). Therefore, the maximum shear stress in the shaft is 103 MPa.

**Example 8-6**

A sign of dimensions 2.0 m x 1.2 m is supported by a hollow circular pole having outer diameter 220 mm and inner diameter 180 mm (Fig. 8-20). The sign is offset 0.5 m from the centerline of the pole and its lower edge is 6.0 m above the ground.

Determine the principal stresses and maximum shear stresses at points A and B at the bases of the pole due to a wind pressure of 2.0 kPa against the sign.

**Fig. 8-26** Example 8-6. Wind pressure against a sign (combined bending, torsion, and shear of the pole).
Solution  

Stress resultants. The wind pressure against the sign produces a resultant force $W$ that acts at the midpoint of the sign (Fig. 8-27a) and is equal to the pressure $p$ times the area $A$ over which it acts:

$$W = pA = (2.0 \text{kPa})(2.0 \text{m} \times 1.2 \text{m}) = 4.8 \text{kN}$$

The line of action of this force is at height $h = 6.6 \text{ m}$ above the ground and at distance $b = 1.5 \text{ m}$ from the centerline of the pole.

The wind force acting on the sign is statically equivalent to a lateral force $W$ and a torque $T$ acting on the pole (Fig. 8-27b). The torque is equal to the force $W$ times the distance $b$:

$$T = Wb = (4.8 \text{kN})(1.5 \text{ m}) = 7.2 \text{kN} \cdot \text{m}$$

The stress resultants at the base of the pole (Fig. 8-27c) consist of a bending moment $M$, a torque $T$, and a shear force $V$. Their magnitudes are

$$M = Wh = (4.8 \text{kN})(6.6 \text{ m}) = 31.68 \text{kN} \cdot \text{m}$$

$$T = 7.2 \text{kN} \cdot \text{m}$$

$$V = W = 4.8 \text{kN}$$

Examination of these stress resultants shows that maximum bending stresses occur at point A and maximum shear stresses at point B. Therefore, A and B are critical points where the stresses should be determined. (Another critical points is diametrically opposite point A, as explained in the Note below).

Stresses at points A and B. The bending moment $M$ produces a tensile stress $\sigma_A$ at point A (Fig. 8-27d) but no stress at point B (which is located on the neutral axis). The stress $\sigma_A$ is obtained from the formula:

$$\sigma_A = \frac{M(d_2/2)}{I}$$

in which $d_2$ is the outer diameter (220 mm) and $I$ is moment of inertia of the cross section. The moment of inertia is

$$I = \frac{\pi}{64}\left(d_2^4 - d_1^4\right) = \frac{\pi}{64}\left[(220 \text{ mm})^4 - (180 \text{ mm})^4\right] = 63.46 \times 10^{-6} \text{ m}^4$$

in which $d_1$ is the inner diameter. Therefore,

$$\sigma_A = \frac{Md_2}{2I} = \frac{(31.68 \text{kN} \cdot \text{m})(220 \text{ mm})}{2(63.46 \times 10^{-6} \text{ m}^4)} = 54.91 \text{ MPa}$$
Fig. 8-27 Solution to Example 8-6.

The torque $T$ produces shear stresses $\tau_1$ at points A and B (Fig. 8-27d). We can calculate these stresses from the torsion formula:

$$\tau_1 = \frac{T (d_2 / 2)}{I_p}$$

in which $I_p$ is the polar moment of inertia:

$$I_p = \frac{\pi}{32} (d_2^4 - d_1^4) = 2I = 126.92 \times 10^{-6} \, m^4$$

Thus,
Finally, we calculate the shear stress at point A and B due to the shear force V. The shear stress at point A is zero, and the shear stress at point B (denoted \( \tau_2 \) in Fig. 8-27d) is obtained from the shear formula for a circular tube (Eq. 5-44 of Section 5.9):

\[
\tau_2 = \frac{4V}{3A} = \left( \frac{r_2^2 + r_1^2}{r_2^2 + r_1^2} \right)
\]

in which \( r_2 \) and \( r_1 \) are the outer and inner radii, respectively, and \( A \) is the cross-sectional area:

\[
r_2 = \frac{d_2}{2} = 110 \text{ mm} \quad r_1 = \frac{d_1}{2} = 90 \text{ mm}
\]

\[
A = \pi(r_2^2 - r_1^2) = 12,570 \text{ mm}^2
\]

Substituting numerical values into Eq. (j), we obtain

\[
\tau_2 = 0.76 \text{ MPa}
\]

All stresses acting at points A and B have now been calculed.

**Stress elements.** The next step is to show these stresses on stress elements (Figs. 8-27e and f). For both elements, the y axis is parallel to the longitudinal axis of the pole and the x axis is horizontal. At point A the stresses acting on the element are

\[
\sigma_x = 0 \quad \sigma_y = \sigma_{\alpha} = 54.91 \text{ MPa} \quad \tau_{xy} = \tau_1 = 6.24 \text{ MPa}
\]

At point B the stresses are

\[
\sigma_x = \sigma_y = 0 \quad \tau_{xy} = \tau_1 + \tau_2 = 6.24 \text{ MPa} + 0.76 \text{ MPa} = 7.00 \text{ MPa}
\]

Since there are no normal stresses acting on the element, point B is in pure shear.

Now that all stresses acting on the stress elements (Figs. 8-27e and f) are known, we can use the equations given in Section 7.3 to determine the principal stresses and maximum shear stresses.

**Principal stresses and maximum shear stresses at point A.** The principal stresses are obtained from Eq. (7-17), which is repeated here:

\[
\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}
\]

Substituting \( \sigma_x = 0, \sigma_y = 54.91 \text{ MPa}, \) and \( \tau_{xy} = 6.24 \text{ MPa}, \) we get

\[
\sigma_{1,2} = 27.5 \text{ MPa} \pm 28.2 \text{ MPa}
\]
or \[ \sigma_1 = 55.7 \text{ MPa} \quad \sigma_2 = -0.7 \text{ MPa} \]

The maximum in-plane shear stresses may be obtained from Eq. (7-25)

\[
\tau_{\text{max}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}
\]

This term was evaluated above, so we see immediately that

\[ \tau_{\text{max}} = 28.2 \text{ MPa} \]